Absolute Maxima and Minima
Optimization
Minimum Average Cost

**Absolute Maxima and Minima**

As explained in detail in the text absolute extrema must always occur at critical values of \( f'(x) \) or at the endpoints of the domain of \( f \). To determine the absolute extrema (maximum or minimum),
1) Find \( f'(x) \)
2) Find the critical numbers by setting \( f'(x) = 0 \) or \( f'(x) = \text{not defined and solving for } x \).
3) Determine if the critical numbers are in the given domain.
4) Evaluate the function \( f \) at the critical numbers and endpoints.
5) Compare the function values and determine the absolute extrema.

**Ex 1** \( f(x) = x^3 - 3x^2 - 24x + 5; \ [-3, 6] \)

\[ f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x - 4)(x + 2) \]

To find the critical number(s), we need to set \( f'(x) = 0 \).

\[ f'(x) = 0 \]
\[ 3(x - 4)(x + 2) = 0 \]
\[ x - 4 = 0 \text{ or } x + 2 = 0 \]
\[ x = 4 \text{ or } -2 \]

The critical numbers are \( x = 4 \) or \( -2 \) and both are in the domain \([-3, 6]\). Next we need to find the corresponding y-coordinates at the endpoints and critical numbers.

\[ f(4) = (4)^3 - 3(4)^2 - 24(4) + 5 = -75; \text{ the smallest } y\text{-value} \]
\[ f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 5 = 33; \text{ the largest } y\text{-value} \]
\[ f(-3) = (-3)^3 - 3(-3)^2 - 24(-3) + 5 = 23 \]
\[ f(6) = (6)^3 - 3(6)^2 - 24(6) + 5 = -31 \]

The absolute maximum, 33, occurs at \( x = -2 \) and the absolute minimum, \(-75\), occurs at \( x = 4 \).

We shall use the graph of \( f(x) \) to check the answers.

**Ex 2** \( f(x) = \frac{x}{x^2 + 2}; [0, 4] \)

\[ f'(x) = \frac{1(x^2 + 2) - x(2x)}{(x^2 + 2)^2} = \frac{2 - x^2}{(x^2 + 2)^2} \]

To find the critical number(s), we need to set \( f'(x) = 0 \) and \( f'(x) = \text{not defined} \).
Since \( x^2 + 2 \) in the denominator is always positive, \( f' \) is always defined.
\[ \frac{2 - x^2}{(x^2 + 2)^2} = 0 \]

Cross multiply
\[ 2 - x^2 = 0 \]
\[ x^2 = 2 \]
\[ x = \pm \sqrt{2} \]

Since \( x = -\sqrt{2} \) is not in the domain, \( x = \sqrt{2} \) is the critical number. Next find the corresponding \( y \)-coordinates.

\[ f\left(\sqrt{2}\right) = \frac{\sqrt{2}}{(\sqrt{2})^2 + 2} = \frac{\sqrt{2}}{4} \approx 0.35 \]

\[ f(0) = 0/(0^2 + 2) = 0 \]
\[ f(4) = 4/(4^2 + 2) = 4/18 = 2/9 \approx 0.22 \]

The absolute maximum, \( \frac{\sqrt{2}}{4} \), occurs at \( x = \sqrt{2} \) and the absolute minimum, 0, occurs at \( x = 0 \).

We use the graph of \( f(x) \) to check the answers.

![Graph of f(x)](image)

**Ex 3** \( f(x) = (x^2 + 18)^{2/3}; [-3, 3] \)

\[ f'(x) = \frac{2}{3} \left( x^2 + 18 \right)^{-1/3} (2x) = \frac{4x}{3 \left( x^2 + 18 \right)^{1/3}} \]

To find the critical number(s), we need to set \( f'(x) = 0 \) and \( f'(x) \) is not defined.

\[ f'(x) = 0 \]
\[ \frac{4x}{3 \left( x^2 + 18 \right)^{1/3}} = 0 \]

Cross multiply and solve.
\[ 4x = 0 \]
\[ x = 0 \]
\[ f'(x) \text{ is not defined } \Rightarrow x^2 + 18 = 0 \Rightarrow x^2 = -18, \text{ no solution.} \]

So, the critical number is \( x = 0 \). Now find the corresponding \( y \)-coordinates.

\[ f(0) = ((0)^2 + 18)^{2/3} = 18^{2/3} \approx 6.87; \text{ the smallest value} \]
\[ f(-3) = ((-3)^2 + 18)^{2/3} = 27^{2/3} = 9; \text{ the largest value} \]
\[ f(3) = ((3)^2 + 18)^{2/3} = 27^{2/3} = 9; \text{ the largest value} \]

The absolute maximum, 9, occurs at \( x = -3 \) and \( x = 3 \). The absolute minimum, \( 18^{2/3} \), occurs at \( x = 0 \).

We shall use the graph of \( f(x) \) to check the answers.
Ex 4 \( f(x) = 2x^3 - 3x^2 - 12x + 24 \), (A) domain = \([-3, 4]\), (B) domain = \([-2, 3]\), (C) domain = \([-2, 1]\).

First we need to find the critical numbers.
\[
f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)
\]
\( x = 2 \) or \( x = -1 \)

\[
f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 24 = 4
\]
\[
f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 24 = 31
\]

(A) domain = \([-3, 4]\)
\[
f(-3) = 2(-3)^3 - 3(-3)^2 - 12(-3) + 24 = -21
\]
\[
f(4) = 2(4)^3 - 3(4)^2 - 12(4) + 24 = 56
\]

Since both critical numbers are within the domain, the absolute maximum point is at \((4, 56)\) and minimum point is at \((-3, -21)\).

(B) domain = \([-2, 3]\)
\[
f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 24 = 20
\]
\[
f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 24 = 15
\]

Since both critical numbers are within the domain, the absolute maximum point is at \((-1, 31)\) and minimum point is at \((2, 4)\).

(C) domain = \([-2, 1]\)
Since both critical numbers are the endpoints of the domain, the absolute maximum point is at \((-1, 31)\) and minimum point is at \((2, 4)\).

In conclusion, the absolute extrema are relative to the given domain.

Ex 5 \( f(x) = x^4 - 18x^2 + 32 \), (A) domain = \([-4, 4]\), (B) domain = \([-1, 1]\), (C) domain = \([1, 3]\).

First we need to find the critical numbers.
\[
f'(x) = 4x^3 - 36x = 4x(x^2 - 9) = 4x(x - 3)(x + 3)
\]
\( x = 0, x = -3, \) or \( x = 3 \)

\[
f(-3) = (-3)^4 - 18(-3)^2 + 32 = -49
\]
\[
f(0) = (0)^4 - 18(0)^2 + 32 = 32
\]
\[
f(3) = (3)^4 - 18(3)^2 + 32 = -49
\]

(A) domain = \([-4, 4]\)
\[
f(-4) = (-4)^4 - 18(-4)^2 + 32 = 0
\]
\[
f(4) = (4)^4 - 18(4)^2 + 32 = 0
\]

Since all critical numbers are within the domain, the absolute maximum point is at \((0, 32)\) and minimum points are at \((\pm3, -49)\).

(B) domain = \([-1, 1]\)
\[
f(-1) = (-1)^4 - 18(-1)^2 + 32 = 15
\]
\[
f(1) = (1)^4 - 18(1)^2 + 32 = 15
\]

Since critical number at \( x = 0 \) is within the domain, the absolute maximum point is at \((0, 32)\) and minimum point is at \((\pm1, 15)\).
(C) domain $=[1,3]$

$f(1) = (1)^4 - 18(1)^2 + 32 = 15$

$f(3) = (3)^4 - 18(3)^2 + 32 = -49$

Since critical number at $x = 3$ is also the endpoint of the domain, the absolute maximum point is at $(1, 15)$ and minimum point is at $(3, -49)$.

**Ex 6** from online homework:

Find the absolute maximum and the absolute minimum, if either exists, for the function $f(x) = x^4 - 8x^3 + 4$ on the given interval. (If a value does not exist, type the letter N.)

(i) interval $[-1, 3]$

<table>
<thead>
<tr>
<th>Absolute Maximum</th>
<th>Absolute Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>-428</td>
</tr>
</tbody>
</table>

(ii) interval $[-1, 5]$

<table>
<thead>
<tr>
<th>Absolute Maximum</th>
<th>Absolute Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>-371</td>
</tr>
</tbody>
</table>

(iii) interval $[0, 5]$

<table>
<thead>
<tr>
<th>Absolute Maximum</th>
<th>Absolute Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-371</td>
</tr>
</tbody>
</table>

First, the computer expects only the $y$-coordinates, not in a point form.

To find the absolute maximum and absolute minimum of the function, you need to find the critical numbers first. Then compare the $y$-coordinates of the critical numbers (values) with those at the endpoints of the domain.

$f'(x) = 4x^3 - 24x^2 + 2 = 4x^2(2x - 6)$

$f'(x) = 0$ gives $x = 0$ and $x = 6$

(i) Since the domain is $[-1, 8]$, you need to find the $y$-coordinates at $x = -1, 0, 6, 8$ and compare.

- $f(-1) = 13$
- $f(0) = 4$
- $f(6) = -428$
- $f(8) = 4$

So the absolute maximum function value is 13 and absolute minimum function value is -428. Similarly for parts (ii) and (iii).

**Ex 7** Find the absolute minimum value on $[0, \infty)$ for $f(x) = (2 - x)(x + 1)^2$.

To find $f'(x)$, be sure to use the product rule.

$f'(x) = (-1)(x+1)^2 + (2-x)(2)(x+1) = (x+1)[-x+2(2-x)] = (x+1)[3-3x] = 3(x+1)(1-x)$

$x = -1$ or $x = 1$

The critical number $x = 1$ is within the domain $[0, \infty)$.

$f(1) = 4$

$f(0) = 2$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (2 - x)(x+1)^2 = -\infty$$

There is no absolute minimum on $[0, \infty)$.

**Ex 8** Find the absolute maximum value on $(0, \infty)$ for $f(x) = 20 - 4x - \frac{250}{x^2}$.

Since the domain is not a closed interval, you need to find the limiting value of $f$.  

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(20 - 4x - \frac{250}{x^2}\right) = -\infty \quad \text{because as } x \to 0, \quad \frac{1}{x} \to \pm\infty \quad \text{and} \quad \frac{1}{x^2} \to 0.
\]

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(20 - 4x - \frac{250}{x^2}\right) = -\infty \quad \text{because as } x \to \infty, \quad -4x \to -\infty \quad \text{and} \quad \frac{1}{x^2} \to 0.
\]

\[
f(x) = 20 - 4x - \frac{250}{x^2} = 20 - 4x - 250x^{-2}
\]

\[
f'(x) = -4 + 500x^{-3} = -4 + \frac{500}{x^3} = \frac{-4x^3 + 500}{x^3} = \frac{-4(x^3 - 125)}{x^3}
\]

\[
x = 5 \quad \text{or} \quad x = 0
\]

But \(x = 0\) is the vertical asymptote and \(\lim_{x \to 0} f(x) = -\infty\)

\[
f(5) = 20 - 4(5) - \frac{250}{(5)^2} = -10
\]

The absolute maximum point is at \((5, -10)\) and the absolute maximum is \(-10\).

Now you can use the TI to check the graph:

Ex 9 \(f(x) = \ln(x^2 e^x)\) on \((0, \infty)\)

Since it is an open interval, we cannot evaluate the function at \(x = 0\) nor at \(x = \infty\). We have to discuss the limits. It will be easier to apply the laws of logs to rewrite the function. Don’t forget that \(\lim_{x \to -\infty} \ln x = -\infty\) and \(\lim_{x \to \infty} \ln x = \infty\).

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} \ln(x^2 e^x) = \lim_{x \to 0} 2\ln x + x = -\infty
\]

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \ln(x^2 e^x) = \lim_{x \to \infty} 2\ln x + x = \infty
\]

The answer is not zero. Remember, \(x \to \infty\) means \(x\) is increasing without bound. We shall use a value to check out the answer. Let \(x = 1000000\).

\[
f(1000000) = 2\ln 1000000 - 1000000 = -999972.369
\]

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \ln(x^2 e^x) = \lim_{x \to \infty} 2\ln x - x = -\infty \quad \text{because } x \to \infty \quad \text{much faster than } \ln x.
\]

\[
f(x) = 2\ln x - x
\]

\[
f'(x) = \frac{2}{x} - 1
\]

\[
f(2) = \ln(2^2 e^2) \approx -0.61 \quad \Rightarrow \quad \text{absolute maximum}
\]
Optimization

To maximize or minimize a function in an application problem is to find the absolute maximum or absolute minimum of the function. We follow the procedure from the previous section.

Ex 1 Find the dimensions of the rectangular field of maximum area that can be made from 200 m. of fencing material (This fence has four sides).

Perimeter of the rectangular field is 200 m. Let \( x \) m be the width.

\[
2\times \text{ length } + 2 \times \text{ width } = 200 \\
2 \times \text{ length } + 2x = 200 \\
2 \times \text{ length } = 200 - 2x \\
\text{ length } = 100 - x 
\]

Since you have to maximize the area, area is the function to be differentiated. The domain is \([0, 100]\).

\[
\text{ area } = \text{ length } \times \text{ width} \\
\text{ f(x) } = (100 - x) x = 100x - x^2 \\
\text{ f'(x) } = 100 - 2x = 0 \Rightarrow x = 50 
\]

Next we need to find the y-coordinates.

\[
\text{ f(50) } = 100(50) - (50)^2 = 2500, \text{ the largest value} \\
\text{ f(0) } = 100(0) - (0)^2 = 0 \\
\text{ f(100) } = 100(100) - (100)^2 = 0 
\]

The absolute maximum, 2500, occurs at \( x = 50 \). The dimensions of the rectangular field are 50 m by 50 m. The largest area is 2500 \( m^2 \).

Ex 2 A fence must be built to enclose a rectangular area of 20,000 \( ft^2 \). Fencing material costs $3 per foot for the two sides facing north and south, and $6 per foot for the other two sides. Find the cost of the least expensive fence.

For this geometry problem, the variables are the length and width of the rectangular area. Since the area of 20,000 \( ft^2 \) is given, use the information to find the width and length. Let \( x \) ft be the width.

\[
\text{ length } \times \text{ width } = 20,000 \\
\text{ length } = 20,000 / x 
\]

\[
\frac{20000}{x} \text{ ft } \times \$3 / \text{ft} \\
x \text{ ft } \times \$5 / \text{ft} \\
\frac{20000}{x} \text{ ft } \times \$3 / \text{ft} 
\]

Since you have to minimize the cost of the least expensive fence, you need to set up the cost function. The domain is \((0, \infty)\).

\[
\text{ cost } = \text{ ($3) length } + \text{ ($3) width } + \text{ ($6) width } + \text{ ($6) width} \\
\text{ f(x) } = 6 \left( \frac{20,000}{x} \right) + 12(x) = 120,000/x + 12x = 120,000x^{-1} + 12x 
\]
\[ f'(x) = -120000x^2 + 12 = -\frac{120000}{x^2} + 12 = 0 \]

\[ \frac{120000}{x^2} = 12 \]

\[ 12x^2 = 120000 \]

\[ x^2 = 10000, x = \pm \sqrt{10000}, x = 100 \]

Next we need to find the y-coordinates.

\[ f(100) = 120,000/(100) + 12(100) = 2400, \] the smallest value because

\[ \lim_{x \to 0} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty \]

The absolute minimum, 2400, occurs at \( x = 100 \). The cost of the least expensive fence is $2400. Dimensions are 200 ft by 100 ft.

To graph this function is a bit tricky. If you don’t set the window correctly, you will have the wrong graph plotted and draw the wrong conclusion. Set the window with \( X_{\text{min}} = 0, X_{\text{max}} = 200, Y_{\text{min}} = 0, \) and \( Y_{\text{max}} = 7000. \)

**Ex 3** A local club is arranging a charter flight to Hawaii. The cost of the trip is $425 each for 75 passengers, with a refund of $5 per passenger for each passenger in excess of 75. a) Find the number of passengers that will maximize the revenue received from the flight. b) Find the maximum revenue.

First, you need to find the cost of the trip if the number of passengers exceeds 75. Let \( x \) be the number of passengers, \( x \geq 75 \). The refund is $5(\( x - 75 \)). So, the cost of the trip for each is $425 - 5(x - 75) = $800 - 5x). You need to maximize the revenue function. The domain is \([75, 160]\).

Revenue = number of passengers * cost

\[ f(x) = x(800 - 5x) = 800x - 5x^2 \]

\[ f'(x) = 800 - 10x = 10(80 - x) = 0, x = 80 \]

\[ f(80) = 800(80) - 5(80)^2 = 32000; \quad \text{the largest value} \]

\[ f(75) = 800(75) - 5(75)^2 = 31875 \]

\[ f(160) = 800(160) - 5(160)^2 = 0 \]

So, the absolute maximum, 32000, occurs at \( x = 80 \). The maximum revenue is $32,000 with 80 passengers.

**Ex 4** A company manufactures and sells \( x \) digital cameras per week. The weekly price-demand and cost equations are, respectively, \( p = 400 - 0.4x \) and \( C(x) = 2000 + 160x \). (A) What price should the company charge for the cameras, and how many cameras should be produced to maximize the weekly revenue? What is the maximum revenue? (B) What is the maximum weekly profit? How much should the company charge for the cameras, and how many cameras should be produced to realize the maximum weekly profit?

(A) Revenue function = \( R(x) = xp = 400x - 0.4x^2 \)

\[ R'(x) = 400 - 0.8x = 0 \]
x = 400/0.8 = 500
Since \( R(x) \) is a quadratic function and its graph is a parabola opening downward. So, the critical number gives the absolute maximum.
\[
R(500) = 400(500) - 0.4(500)^2 = 100,000
\]
500 cameras should be produced to give a maximum weekly revenue of $100,000.
\[ p = 400 - 0.4(500) = 200 \]
The company should charge $200 per camera.

(B) Profit function \( P(x) = R(x) - C(x) = 400x - 0.4x^2 - (2000 + 160x) = -0.4x^2 + 240x - 2000 \)
\[
P'(x) = -0.8x + 240 = 0
\]
x = 240/0.8 = 300

Since \( P(x) \) is a quadratic function and its graph is a parabola opening downward. So, the critical number gives the absolute maximum.
\[
P(300) = -0.4(300)^2 + 240(300) - 2000 = 34,000
\]
300 cameras should be sold to give a maximum weekly profit of $34,000.
\[ p = 400 - 0.4(300) = 280 \]
The company should charge $280 per camera.

Ex 5: from online chapter homework

A company manufactures and sells \( x \) video phones per week. The weekly price-demand and cost equations are \( p = 400 - 0.5x \) and \( C(x) = 305x + 10000 \).

(i) To maximize their weekly revenue, the company should charge \( \$200 \) per phone and produce \( 400 \) phones. The maximum weekly revenue is \( \$80000 \).

(ii) The maximum weekly profit is \( \$5487.5 \) and occurs when the company produces \( 95 \) phones per week at a price of \( \$352.5 \) per phone.

Parts (i) and (ii) are different. Part (i) is about revenue = \( xp \) and part (ii) is about profit = revenue - cost.

(i) \( R(x) = xp = x(400 - 0.5x) = 400x - 0.5x^2 \)
\[
R'(x) = 400 - x
\]
\[
R''(x) = 0 \text{ gives } x = 400
\]
R''(x) = -1 < 0 means graph of \( R \) is concave downward.
So, \( R \) is a maximum at \( x = 400 \) phones
\[
\text{price} = p = 400 - 0.5(400) = 200
\]
\[
\text{max revenue} = R(400) = 400(400) - 0.5(400)^2 = \$80000
\]

(ii) \( P(x) = R(x) - C(x) = 400x - 0.5x^2 - 305x - 10000 = -0.5x^2 + 95x - 10000 \)
\[
P'(x) = -x + 95
\]
P''(x) = -1 means graph of \( P \) is concave downward
\[
\text{maximum profit at } x = 95 \text{ phones}
\]
\[
\text{max profit} = P(95) = ... \approx -5487.5 \text{ which is a negative profit. In fact the company is losing money. If this is the max profit, then the company is losing more for other values of } x.
\]
\[
\text{price} = p = 400 - 0.5(95) = \$352.5
\]

Ex 6 A parcel delivery service will deliver a package only if the length plus girth (distance around) does not exceed 108 inches. (A) Find the dimensions of a rectangular box with square ends that satisfies the delivery service’s restriction and has maximum volume. What is the maximum volume? (B) Find the dimensions (radius and height) of a cylindrical container that meets the delivery service’s requirement and has maximum volume. What is the maximum volume?
(A) First you need to identify the variables and the function to be maximized in the problem. The variables are the length, width, and the height. You shall maximize the volume of the box. Since the length plus the girth should be no more than 108 inches and the base is a square, you use this information to set up equations for the variables.

Let width of the base = \( x \) inches.
Length of the base = width = \( x \) inches.
Height + 2(length) + 2(width) = 108
Height + 2\( x \) + 2\( x \) = 108
Height = 108 – \( 4x \)

Now you set up the function for the volume of the box.
Volume = width * length * height
\[ f(x) = x \times x \times (108 - 4x) = x^2(108 - 4x) \]
\[ f(x) = 108x^2 - 4x^3 \]

Next we also need to find the domain of the volume function. Since all measurements must be greater than or equal to zero, 0 is the smallest value of \( x \). If length = 0, then 108 – 4\( x \) = 0 or \( x \) = 27. Therefore, the domain is [0, 27]. Now we will find the critical numbers.

\[ f'(x) = 216x - 12x^2 = 12x(18 - x) = 0 \Rightarrow x = 0 \text{ or } x = 18 \]

\[ f(0) = 108(0)^2 - 4(0)^3 = 0 \]
\[ f(18) = 108(18)^2 - 4(18)^3 = 11664; \text{ the largest value} \]
\[ f(27) = 108(27)^2 - 4(27)^3 = 0 \]

The absolute maximum, 11664, occurs at \( x = 18 \). Therefore, the box has a width of 18 inches, length of 18 inches, and the height of 108 – 4(18) or 36 inches. The maximum volume is 11664 cubic inches.

We use the graph to check the answers.

(B) Let \( r \) be the radius and \( h \) be the height of a cylindrical container that meets the delivery service’s requirement.

\[ \text{Girth} = h + 2\pi r = 108, \ h = 108 - 2\pi r \]
\[ V = \pi r^2 h = \pi r^2(108 - 2\pi r) = 108\pi r^2 - 2\pi^2 r^3 \]
\[ V' = 216\pi r - 6\pi^2 r^2 \]
\[ V' = 0 \]
\[ 6\pi r(36 - \pi r) = 0 \]
\( r = 0 \) or \( r = \frac{36}{\pi} \)

We will use \( V'' \) to check the concavity at the critical numbers.

\[
V'' = 216\pi - 12\pi^2 r
\]

\[
V'' \left( \frac{36}{\pi} \right) = 216\pi - 12\pi^2 \left( \frac{36}{\pi} \right) = -216\pi < 0
\]

\( V \) has a maximum at \( r = \frac{36}{\pi} \approx 11.5 \).

\( h = 108 - 2\pi r = 108 - 2\pi \left[ \frac{36}{\pi} \right] = 36 \)

\[
V = \pi \left( \frac{36}{\pi} \right)^2 (36) = \frac{46656}{\pi} \approx 14851 \text{ cubic inches}
\]

So, the cylinder with radius of 11.5 inches and height of 36 inches has a maximum volume of 14851 cubic inches.

You may use TI to check the beautiful answers but not to replace the work:

Ex 7: From online chapter home work.

Oil is being piped from an offshore oil well to a refinery on land. The shoreline is considered straight, the well is 5 miles out to sea (i.e., at a right angle to the shore), the refinery is 7 miles along the coast, underwater pipe costs $240000/mile to construct, but pipeline only costs $150000/mile on land. Given this, how far down the coast should the pipeline be when it reaches land, if cost is to be minimized? (0 miles' would mean build straight to shore, then make a right angle to the refinery, '7 miles' would mean to build the pipe directly to the refinery).
Let \( x \) miles be the distance as shown in the diagram. The length of the pipeline on land is \((7 - x)\) miles. The length of the pipeline underwater is \(\sqrt{25 + x^2} \) miles. So, the cost \( C \) is the cost to build the pipeline on land and the cost to build the pipeline underwater.

\[
C = 150000(7 - x) + 240000\sqrt{x^2 + 25}
\]

\[
C' = -150000 + 240000 \left( \frac{1}{2} \right) (x^2 + 25)^{-1/2} (2x) = -150000 + \frac{240000x}{\sqrt{x^2 + 25}}
\]

\[
C' = 0 \rightarrow \frac{240000x}{\sqrt{x^2 + 25}} = 150000
\]

\[
240000x = 150000\sqrt{x^2 + 25}
\]

\[
8x = 5\sqrt{x^2 + 25} \quad \text{square both sides}
\]

\[
64x^2 = 25(x^2 + 25)
\]

\[
64x^2 = 25x^2 + 625
\]

\[
39x^2 = 625
\]

\[
x^2 = \frac{625}{39} \rightarrow x = \pm \sqrt{\frac{625}{39}} = \pm \frac{25}{\sqrt{39}} \rightarrow x = \frac{25\sqrt{39}}{39} \approx 4.0
\]

It should be about 4 miles.

You still need to use \( C'' \) to verify the concavity and put the answers in the correct form.

**Ex 8** From online home work.

**Homework**  Online Chapter 4 Home Work

**Exercises**

A rectangular enclosure of 16400 m\(^2\) is to be built. One side of the enclosure will be flush against a long building and will not require any fencing. The side opposite the building will be constructed in a manner that will cost $2.50/m, while the other two sides will cost $3.00/m. Find the dimensions that will minimize cost.

a. \(20\sqrt{123}\) by \(20\sqrt{41/3}\)

b. \(10\sqrt{205/3}\) by \(8\sqrt{615}\)

c. \(5\sqrt{205/3}\) by \(48\sqrt{615}\)

d. \(20\sqrt{41}\) by \(20\sqrt{41}\)

Let \( y \) be the side facing the building and \( x \) be the other two sides. So, using the area, we can express \( y \) in terms of \( x \).
xy = 16400 \rightarrow y = \frac{16400}{x}

Now we need to set up the cost function and minimize it.

\[ C = 3x + 3x + 2.5y = 6x + 2.5 \left( \frac{16400}{x} \right) = 6x + \frac{41000}{x} \]

To minimize \( C \), we need to find \( C'(x) \) and set \( C'(x) = 0 \) to solve for \( x \).

\[ C' = 6 - \frac{41000}{x^2} \]

\[ C'(x) = 0 \Rightarrow 6 - \frac{41000}{x^2} = 0 \]

\[ 6 = \frac{41000}{x^2} \]

\[ 6x^2 = 41000 \]

\[ x^2 = \frac{41000}{6} \]

\[ x = \pm \sqrt{\frac{41000}{6}} = \pm \sqrt{\frac{20500}{3}} = \pm 10 \sqrt{\frac{205}{3}} \]

Since \( x \) cannot be negative, \( x = 10 \sqrt{\frac{205}{3}} = 82.663978 \approx 82.7 \).

Now how do we know it will give a minimum of \( C \)? Use \( C'' \) to check out the concavity.

\[ C'' = 6 - \frac{41000}{x^2} = 6 - 41000x^{-2} \]

\[ C'' = 12(41000)x^{-3} \]

\[ C''(10 \sqrt{\frac{205}{3}}) = 12(41000) \left( 10 \sqrt{\frac{205}{3}} \right)^{-3} > 0 \]

So the graph is concave upward at \( x = 10 \sqrt{\frac{205}{3}} \). It is a minimum.

To find \( y \), we substitute the \( x \)-value into the area.

\[ y = \frac{16400}{x} = 16400 \left( \frac{1}{10 \sqrt{\frac{205}{3}}} \right) = \frac{16400 \sqrt{3(205)}}{205} = 8 \sqrt{615} = 198.3935 \approx 198.4 \]

Dimensions are \( 10 \sqrt{\frac{205}{3}} \) by \( 8 \sqrt{615} \) m².

**Minimum Average Cost**

**Ex 1** Find the absolute minimum of the average cost function, \( \bar{C}(x) = \frac{C(x)}{x} \) for the cost function \( C(x) = 10 + 20x^{1/2} + 16x^{3/2} \).

\[ \bar{C}(x) = \frac{10 + 20x^{1/2} + 16x^{3/2}}{x} = 10x^{-1} + 20x^{-1/2} + 16x^{1/2} \]

\[ \bar{C}'(x) = -10x^{-2} - 10x^{-3/2} + 8x^{-1/2} \]

Use TI to solve for \( x \).
The critical number is $x = 2.1104448$. From the graph of $\overline{C}'(x)$, we can set up the sign graph.

\[
\begin{array}{cccc}
  - & - & - & - \\
  2.11 \\
  + & + & + & + \\
\end{array}
\]

$\overline{C}(x)$ has a local minimum at $x = 2.11$.

$$\overline{C}(2.1104448) = 10(2.1104448)^{-1} + 20(2.1104448)^{-1/2} + 16(2.1104448)^{1/2} = 41.74924647 \approx 41.75$$

To determine the domain, $x$ is the number of items sold in the cost function. Since average cost is the total cost divided by $x$, the domain is $(0, \infty)$. To find the y-coordinates at the endpoints, we need to evaluate the limiting values at the endpoints.

$$\lim_{{x \to 0^+}} \frac{10x^{-1} + 20x^{-1/2} + 16x^{1/2}}{x} = \infty \quad \text{and} \quad \lim_{{x \to \infty}} \frac{10x^{-1} + 20x^{-1/2} + 16x^{1/2}}{x} = \infty$$

Therefore, the absolute minimum, 41.75, occurs at $x \approx 2$. To check the answer using TI, enter the average cost function and do the following:

\[
\begin{array}{ccc}
  \text{Ex 2} & \text{Minimum average cost} & = \overline{C}(x) = \frac{C(x)}{x}, \text{ where } C(x) = 400x - 200\ln x + 700; x \geq 3. \\
  \overline{C}(x) & = \frac{400x - 200\ln x + 700}{x} & \text{split into fractions} \\
  & = 400 - 200\frac{\ln x}{x} + 700x^{-1} \\
  \overline{C}'(x) & = -200\left(\frac{1}{x^2}\right)x - (\ln x)(1) \\
  & = -200\left(\frac{1}{x^2}\right)x - 700x^{-2} \\
  & = -200\frac{1 - \ln x}{x^2} - 700x^{-2} \\
  & = -900 + 200\ln x \\
  \overline{C}'(x) & = 0 \rightarrow -900 + 200\ln x = 0 \\
  \ln x & = \frac{9}{2}, x = e^{9/2} = 90.01713 \approx 90 \\
\end{array}
\]

To make sure the average cost is a minimum, we use the second derivative test. A good exercise for quotient rule.
\[ \bar{C}''(x) = \frac{200 \left( \frac{1}{x} \right) x^2 - (-900 + 200 \ln x)(2x)}{(x^2)^2} \]
\[ = \frac{200x + 1800x - 400x \ln x}{x^4} \]
\[ = \frac{2000 - 400 \ln x}{x^4} \]
\[ \bar{C}''(e^{4.5}) = \frac{2000 - 400 \ln e^{4.5}}{(e^{4.5})^4} = \frac{200}{e^{31.5}} > 0 \]

So, average cost has a minimum at \( x = e^{4.5} \).

\[ \bar{C}(e^{4.5}) = \frac{400e^{4.5} - 200 \ln e^{4.5} + 700}{e^{4.5}} \]
\[ = \frac{400e^{4.5} - 700}{e^{4.5}} = 400 - 200 e^{4.5} \approx 397.78 \]

The minimum average cost is about $397.78.